Semisimple Modules, Jacobson Radicals, and Related Results

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1 Introduction

These notes discuss basic concepts in ring and module theory, focusing on *semisimple* modules and the *Jacobson radical* of a ring. We give definitions, some standard examples, and key theorems such as Artin–Wedderburn. We also discuss why a ring A with zero Jacobson radical (i.e. $J(A) = 0$) often exhibits "complete reducibility" of its modules.

Throughout, let A be a (possibly noncommutative) ring (often we assume A is a finite-dimensional algebra over a field, or at least Artinian) and let A-modules be left modules unless otherwise specified.

2 Historical Notes

The study of *semisimple* rings and modules has roots in the early 20th century, closely tied to the work of **Joseph Henry Maclagan Wedderburn** (around 1908) on the structure of finite-dimensional algebras over a field. Wedderburn showed that if an algebra A is semisimple (in modern language), then it is isomorphic to a finite direct product of matrix algebras over division rings. This result formed the cornerstone of what became known as the **Wedderburn–Artin theory**, after **Emil Artin** generalized these ideas to more general Artinian rings.

The *Jacobson radical* is named after **Nathan Jacobson**, who in the mid-20th century introduced a unifying definition of the radical as the intersection of all maximal left ideals (or right ideals, for rings with identity). Jacobson's approach systematically explained why certain ideals behave "like nilpotent elements" in the sense of vanishing on all simple modules. His work built on earlier ideas about "maximal nil ideals" or "quasi-regular elements" but gave them a coherent modern framework.

Subsequently, mathematicians such as **R. Baer**, **J. Levitzki**, **A. G. Kurosh**, and **S. A. Amitsur** extended Jacobson's concepts into a broad *radical theory*, defining many other radicals (prime radical, Baer radical, Levitzki radical, etc.) and studying their behavior in various classes of rings. These developments showed how the notion of a radical could be abstracted and applied in more general algebraic settings, including non-Artinian rings, rings without identity, and even certain rings in universal algebra. Nevertheless, the *Jacobson radical* remains one of the most central and influential of these constructions, due to its decisive role in characterizing semisimple rings: a (left) Artinian ring A is semisimple if and only if $J(A) = 0$.

3 Semisimple Modules

Definition 3.1 (Simple and Semisimple Modules)**.** Let A be a ring. An A-module S is called *simple* if its only submodules are 0 and S itself.

An A-module M is called *semisimple* (or *completely reducible*) if it can be written as a direct sum of simple modules:

$$M \;\cong\; S_1 \oplus S_2 \oplus \cdots \oplus S_n \quad \text{for some simple } A\text{-modules } S_i.$$

Example 3.2. Over a field k, a vector space is simple as a module over itself precisely when it is 1-dimensional. Thus a k-vector space is semisimple (as a module over k) if and only if it is a direct sum of 1-dimensional subspaces (i.e. it is just any vector space). So in this case, *every* module is semisimple.

Proposition 3.3 (Basic Properties of Semisimple Modules)**.** *Let M be a semisimple A-module. Then:*

1. *Every submodule $U \leq M$ is also semisimple.*
2. *Every quotient M/U is semisimple.*
3. *M is semisimple if and only if every submodule of M is a direct summand. In other words, for any $U \leq M$, there exists a complement $W \leq M$ such that $M = U \oplus W$.*

Sketch of Proof. If $M \cong \bigoplus_i S_i$ where each S_i is simple, any submodule U is a direct sum of some of these simple summands (one can prove this using a composition series argument or by induction). Thus U is semisimple, and the quotient M/U is also a direct sum of simple modules. For the statement about direct summands, the standard approach is to use the fact that semisimple modules have projective covers, or to invoke Schur's Lemma in the setting of finite dimensional algebras, etc. □

4 Jacobson Radical

Definition 4.1 (Jacobson Radical)**.** Let A be a ring. The *Jacobson radical* of A, denoted $J(A)$, is the intersection of all maximal left ideals of A. Equivalently (for rings with identity), it is the intersection of all maximal right ideals of A. One can also characterize $J(A)$ as the set of $r \in A$ that act nilpotently on *every* simple left A-module.

In symbols:

$$J(A) \;=\; \bigcap_{\substack{M \subset A \\ M \text{ max left ideal}}} M \;=\; \bigcap_{\substack{N \subset A \\ N \text{ max right ideal}}} N.$$

Remark 4.2. Intuitively, $J(A)$ is the "largest nilpotent ideal" inside A in many senses. For finite dimensional algebras, it coincides with the *radical of the algebra* used in the Artin–Wedderburn decomposition.

Proposition 4.3. *If A is a semisimple ring (i.e. semisimple as a left module over itself), then $J(A) = 0$.*

Sketch of Proof. If A is semisimple as a left module, then any element of $J(A)$ must act as zero on every simple A-module. In particular, consider the regular representation A_A (the module A acting on itself by left multiplication). If A is semisimple, A_A is a direct sum of simple submodules. Since $J(A)$ annihilates every simple submodule, it must be zero in A itself. □

5 Characterizations of Semisimple Rings

Recall that a ring A is called *semisimple* (on one side, say left) if it is semisimple as a left A-module. Over an Artinian (or finite dimensional) ring, left semisimplicity, right semisimplicity, and two-sided semisimplicity all coincide.

Theorem 5.1 (Classical Criterion for Semisimplicity)**.** *Let A be a ring which is Artinian (or at least satisfies the descending chain condition on ideals). Then the following are equivalent:*

1. *A is semisimple (i.e. A_A is a direct sum of simple modules).*
2. *$J(A) = 0$.*
3. *Every A-module is semisimple.*

Idea of Proof. The heart of the proof is the observation that if $J(A) = 0$, then A can be embedded into a direct product (or direct sum) of simple rings (via the map $a \mapsto (a + M_1, \dots, a + M_n)$ where M_i are maximal ideals whose intersection is $\{0\}$). Such a product is semisimple. Hence A itself inherits semisimplicity. The other directions are handled by the standard theory of radicals and composition series arguments. □

Remark 5.2. If A is a finite dimensional algebra over a field k, then A is Artinian. So all of the above equivalences apply. In this case, the Artin–Wedderburn theorem states that

$$A \;\cong\; M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_t}(D_t),$$

where each D_i is a finite dimensional division algebra over k.

6 Embedding into a Direct Sum of Simple Modules

A construction often used in these proofs is the following:

- Pick maximal right ideals $M_1, \ldots, M_n$ such that $M_1 \cap \cdots \cap M_n = \{0\}$.
- Define a map of left A-modules:

$$T : A \longrightarrow \bigoplus_{i=1}^{n} A/M_i, \quad a \mapsto (a + M_1, a + M_2, \ldots, a + M_n).$$

- The kernel of T is exactly $M_1 \cap \cdots \cap M_n$, which is 0 by choice of M_i. Hence T is injective.
- Since each A/M_i is a simple left A-module (quotient by a maximal ideal), the direct sum $\bigoplus_{i=1}^{n} A/M_i$ is semisimple.
- Consequently, A embeds into a semisimple left A-module, so if also $J(A) = 0$, we deduce A itself is semisimple.

7 Complements and Short Exact Sequences

One important property of semisimple modules V is that every submodule $U \leq V$ has a *direct sum complement* W, i.e. $V \cong U \oplus W$. This makes short exact sequences

$$0 \longrightarrow U \longrightarrow V \longrightarrow V/U \longrightarrow 0$$

split. In particular:

Corollary 7.1. *If V is a semisimple A-module, then every submodule U is semisimple and the quotient V/U is semisimple.*

Proof. By definition, V is a direct sum of simple modules. Any submodule U is a sum of some of these simple summands (or possibly zero), so U is semisimple. Its quotient V/U is also a direct sum of the remaining simple summands. □

Example 7.2 (Counterexamples in Non-Semisimple Cases)**.** If V is *not* semisimple, it can fail to have such a direct complement. For instance, over $\mathbb{Z}_4$, the module $\mathbb{Z}_4$ (viewed over itself) is not semisimple because it has nontrivial submodules that are not direct summands (e.g. $2\mathbb{Z}_4$ is a submodule isomorphic to $\mathbb{Z}_2$, but you cannot write $\mathbb{Z}_4$ as a direct sum $\mathbb{Z}_2 \oplus W$ in a way that respects the $\mathbb{Z}_4$-module structure).

8 A Quick Look at the Artin–Wedderburn Theorem

Theorem 8.1 (Artin–Wedderburn)**.** *Let A be a finite dimensional semisimple algebra over a field k. Then A is isomorphic (as an algebra) to a finite direct product of matrix algebras over division algebras over k:*

$$A \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_r}(D_r),$$

where each D_i is a finite dimensional division algebra over k.

This theorem fully classifies finite dimensional semisimple algebras. If k is algebraically closed, each D_i is just k itself, and so

$$A \cong M_{n_1}(k) \times M_{n_2}(k) \times \cdots \times M_{n_r}(k).$$

9 Why Some Proofs are More Involved

The classification and structure theorems for semisimple modules can be proven in several ways. The short proof typically relies on:

- The idea that $J(A) = 0$ forces complete reducibility.
- Splitting of short exact sequences for semisimple modules.
- Wedderburn's theorem for finite dimensional semisimple algebras.

Longer proofs might break these arguments down further or proceed via explicit homological algebra (projective modules, projective covers, etc.) or by carefully constructing complements for each submodule. In a more advanced course, one might discuss injective/projective dimensions, minimal projective resolutions, and so on, which can obscure the simpler direct sum arguments.

10 Further Applications

10.1 Mathematical and Theoretical Applications

Further Radical Theory. The Jacobson radical is only one of many radicals defined in ring theory. Other important radicals include the *prime radical* (or Baer–McCoy radical), the *Baer radical*, and the *Levitzki radical*, among others. Each radical is designed to capture a specific "bad behavior" within a ring (e.g. nilpotency, solvability, etc.). These concepts form the basis of *radical theory*, which has far-reaching consequences in:

- The classification of non-Artinian rings.
- Structure theory of rings without identity.
- Connections to universal algebra and varieties of algebras.

Wedderburn–Artin Theory and Beyond. The Wedderburn–Artin theorem classifies *all* semisimple rings (which are automatically Artinian if unital). Its extension to non-commutative settings and applications to group algebras, matrix rings, and operator algebras are central in:

- **Representation Theory of Finite Groups:** Over a field of characteristic zero, semisimplicity of the group algebra $k[G]$ relates to Maschke's theorem, ensuring all representations decompose into irreducibles.
- **Structure of Division Algebras:** Central division algebras over fields are intimately connected with the Brauer group, an active research area linking algebraic geometry and number theory.

- **Homological Algebra and Derived Categories:** Semisimple algebras have particularly simple homological properties (e.g. projective dimension zero). This facilitates explicit computations in derived categories of modules.

10.2 Applications in Physics and Computer Science

Physics.

- **Quantum Mechanics and Quantum Field Theory:** Many operator algebras arising in quantum mechanics are (or can be reduced to) direct sums of matrix algebras over $\mathbb{C}$. Semisimple Lie algebras (a related but distinct concept in the Lie algebra setting) underlie the classification of particle symmetries and gauge groups in quantum field theories.
- **Symmetry Algebras and Representations:** Physical systems with symmetry often use representation theory to describe states and observables. The complete reducibility (semisimplicity) of certain symmetry algebras simplifies the decomposition of state spaces into irreps (irreducible representations), directly impacting how physical particles or excitations are classified.

Computer Science.

- **Computational Group Theory and Symbolic Algebra Systems:** Algorithms that compute irreducible representations of finite groups often rely on the group algebra's semisimplicity (when the characteristic of the field does not divide the group order). Decomposing modules into direct sums of simples is a fundamental step in these algorithms, which are implemented in software like GAP, SageMath, and Magma.
- **Coding Theory and Cryptography:** Certain error-correcting codes and cryptographic protocols involve modules over finite rings or group algebras. In some cases, understanding the radical and semisimplicity helps classify the structure of these rings, leading to efficient encoding/decoding or security arguments.
- **Quantum Computing:** While not as common as operator algebras in theoretical physics, the study of finite-dimensional C*-algebras (which are direct sums of matrix algebras over $\mathbb{C}$) appears in certain formulations of quantum logic and quantum information. Complete reducibility can simplify how quantum states and gates factor into simpler subsystems.

11 Summary

- **Semisimple modules** are direct sums of simple modules.
- A **ring** A **is semisimple** if A is semisimple as a module over itself (equivalently, it has zero Jacobson radical and is Artinian).
- **Artin–Wedderburn** classifies semisimple algebras as finite products of matrix algebras over division rings.
- **Jacobson radical** $J(A)$ is the intersection of all maximal ideals (one or both sides). If A is semisimple, $J(A) = 0$. Conversely, if $J(A) = 0$ and A is Artinian, then A is semisimple.

- A **semisimple module** M has the property that every short exact sequence $0 \to U \to M \to M/U \to 0$ splits, giving $M \cong U \oplus (M/U)$.
- These concepts and theorems find wide applications, both in further **radical theory, Wedderburn–Artin theory, representation theory** and beyond, and in more **applied contexts** such as *physics, computational group theory, coding theory*, and *quantum computing.*

These results form a cornerstone in the representation theory of finite dimensional algebras and in module theory in general.